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UNIFORM POINTWISE ERGODIC THEOREMS FOR CLASSES OF AVERAGING SETS AND MULTIPARAMETER SUBADDITIVE PROCESSES

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Recently, Bass and Pyke proved a strong law of large numbers for d-dimensional arrays of i.i.d. random variables in which the a.e.-convergence was uniform over a large family of averaging sets. Using different arguments from ergodic theory, we extend this result to multiparameter subadditive processes. Even without the uniformity statement this yields convergence a.e. for more general averaging sequences than those considered by Akcoglu and Krengel.

ergodic theory * multiparameter subadditive process

1. Introduction

For a fixed integer $d \geq 1$ let V denote the set $\{0, 1, 2, \dots\}^d$ and $\mathcal{V}$ the set of all finite non-empty subsets of V. A real valued process $F = \{F_A : A \in \mathcal{V}\}$ on a given probability space $(\Omega, \mathcal{A}, P)$ is called *stationary* if, for any $k \geq 1$, any $A_1, \dots, A_k \in \mathcal{V}$ and any $u \in \mathcal{V}$ the joint distribution of $F_{A_1}, \dots, F_{A_k}$ is the same as that of $F_{A_1+u}, \dots, F_{A_k+u}$. The parameter set of the process may be extended by putting $F_\phi = 0$ and $F_B := F_{B \cap V}$ for any bounded $B \subset [0, \infty]^d$.

For $u = (u_i)$, $v = (v_i) \in \mathbb{R}^d$ we write $[u, v) = \{(w_i): u_i \leq w_i < v_i,\ i = 1, \dots, d\}$ for d-dimensional intervals. Put $e = (1, 1, \dots, 1)$. F is called *subadditive* if it satisfies the following conditions:

(i) F is stationary;
(ii) $F_{A \cup B} \leq F_A + F_B$ holds for disjoint $A, B \in \mathcal{V}$;
(iii) the random variables F_A are integrable; and
(iv) $\gamma(F) := \inf\{\int n^{-d} F_{[0, ne)}\, dP\} > -\infty$.

F is called *superadditive* if $-F$ is subadditive and *additive* if F is super- and subadditive.

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For measurable $B \subset [0, \infty)^d$, $\lambda(B)$ shall denote the Lebesgue measure of B and ∂B the boundary of B. If ρ is the Euclidean distance, $B(\delta) := \{v \in [0, \infty)^d : \rho(v, \partial B) < \delta\}$ is the *δ-annulus* of ∂B. Put $nB = \{nv : v \in B\}$.

Theorem 1. *Suppose $\mathscr{B}$ is a collection of Borel measurable subsets of* $[0, 1]^d$ *such that*

$$r_{\mathscr{B}}(\delta) := \sup\{\lambda(B(\delta)) : B \in \mathscr{B}\} \to 0 \quad \text{as } \delta \to 0. \tag{1}$$

For subadditive F there exists $\bar{f} \in L_1$ with

$$\sup\{|n^{-d} F_{nB} - \lambda(B)\bar{f}| : B \in \mathscr{B}\} \to 0 \quad a.s. \tag{2}$$

as $n \to \infty$.

Clearly, $\bar{f}$ must be the L_1-limit of the averages $\bar{F}_u = (u_1 \ldots u_d)^{-1} F_{[0,u)}$ for $u \to \infty$ (meaning that all coordinates u_i of u tend to ∞). Smythe [9] showed that this limit exists. The almost sure convergence of $n^{-d} F_{[0,ne)}$ is due to Akcoglu and Krengel [1], and their maximal inequality is the key ingredient for the present proof.

Actually, they considered more general averaging sets than squares. Similarly, the theorem above permits a generalization: For $u \in V$ put

$$uB = \{(u_1 v_1, u_2 v_2, \ldots, u_d v_d) : v \in B\}.$$

Let $(u(k))$ be an increasing sequence in V with $u(k) \to \infty$. Then (2) can be replaced by

$$\sup\{|\lambda([0, u(k)))^{-1} F_{u(k)B} - \lambda(B)\bar{f}| : B \in \mathscr{B}\} \to 0 \quad \text{a.s.} \tag{3}$$

We remark that the case of sequences of convex averaging sets has been sketched in the recent monograph of Krengel [7]. Clearly the family $\mathscr{C}$ of convex subsets of $[0, e]$ satisfies $r_{\mathscr{C}}(\delta) \to 0 (\delta \to 0)$, but it is easy to give examples of families $\mathscr{B}$ with $r_{\mathscr{B}}(\delta) \to 0$ containing non-convex sets. For $\mathscr{B} = \mathscr{C}$ the novel point is the uniformity of convergence. In the additive case, Tempel'man [11] considered averages also over some suitable sequences of nonconvex sets; see [7] for details and other references

2. Proof of the theorem

It will be enough to prove the desired convergence for nonnegative superadditive processes, because the additive process F^1 given by $F^1_A = \sum_{v \in A} F_{\{v\}}$ is a difference of two nonnegative additive processes and $-F + F^1$ is nonnegative superadditive. We therefore assume now that F is nonnegative superadditive.

We can also assume that there are commuting measure preserving transformations $\tau_1, \ldots, \tau_d$ of $(\Omega, \mathscr{A}, P)$ such that $F_{A+u} = F_A \circ \tau_u$, where $\tau_u = \tau_1^{u_1} \tau_2^{u_2} \ldots \tau_d^{u_d}$ for $u \in V$. This may be seen by replacing Ω by $\mathbb{R}^{V}$ as in the usual product space representation of a process.

The spatial constant of the superadditive process F is given by

$$\gamma := \gamma(F) = \sup\left\{\int \bar{F}_u \,\mathrm{d}P \colon u \in \mathbb{N}^d\right\}.$$

It is well known and easy to see that this supremum is a limit for $u \to \infty$.

We now need an argument from [4]. Let $\sum_{k,n}$ be the sum of all $F_{[ku,k(u+e))}$ with $0 \leq u_i < [n/k]$ for $i = 1, \ldots, d$, where $[x]$ is the largest integer $\leq x$. For any $\varepsilon > 0$ we may find k with $\int \bar{F}_{ke}\,\mathrm{d}P > \gamma - \varepsilon^2$. The additive mean ergodic theorem implies the convergence in L_1-norm of $k^{-d}[n/k]^{-d}\sum_{k,n}$ to some g_k invariant under τ_{ke} and having integral

$$\int g_k \,\mathrm{d}P = \int \bar{F}_{ke}\,\mathrm{d}P.$$

The proof in [4, p. 41–42] shows that (g_k) is a Cauchy sequence converging in L_1 to some $\bar{f} \in L_1$ invariant under all τ_u, and that $\|\bar{F}_u - \bar{f}\|_1 \to 0 (u \to \infty)$ and $\int \bar{f}\,\mathrm{d}P = \gamma$.

Recall that, for any k, L_2 is the orthogonal direct sum of the space H_k of vectors which are fixed under all τ_i^k $(i = 1, \ldots, d)$, and the closure of the set of sums of the form $\sum_{i=1}^d (a_i \circ \tau_i^k - a_i)$ with $a_i \in L_2$; see [7, Lemma 1.1.3]. The projection $P_k h$ of h on H_k is the limit of $m^{-d}\sum_{u<me} \tau_{ku} h$. If h is close in L_1-norm to $\bar{F}_{ke}$, then $P_k h$ is close to g_k. For large k, g_k is arbitrarily close to $\bar{f}$. Modifying the a_i a little in L_1-norm we can assume that they are bounded. Putting all this together, we find that, for any $\varepsilon > 0$ there exist bounded functions $a_1, a_2, \ldots, a_d$ and a function b with $\|b\|_1 < \varepsilon^2$ such that, for some large enough k,

$$\bar{F}_{ke} = \bar{f} + \sum_{i=1}^{d} (a_i \circ \tau_i^k - a_i) + b. \tag{4}$$

For arbitrarily small $\eta > 0$ there exists a large m with

$$\left| m^{-d} \sum_{u<me} \tau_{ku} \sum_{i=1}^{d} (a_i \circ \tau_i^k - a_i) \right| < \eta \quad \text{a.e.,}$$

because the a_i are bounded and most of the terms cancel each other.

Put $M = km$. It will be convenient to consider a fixed set $B \subset [0,1]^d$ from the family $\mathcal{B}$. The estimates shall show that the convergence is uniform in $\mathcal{B}$.

Let K_n be the set of all $u \in V$ with $[Mu, M(u+e)) \subset nB$ and L_n the set of all $u \in V$ with $[Mu, M(u+e)) \cap nB \neq \emptyset$. C_n (and D_n) denote the union of all intervals $[Mu, M(u+e))$ with $u \in K_n$ (and with $u \in L_n$). We have $C_n \subset nB \subset D_n$ and, by (1),

$$\lim_{n\to\infty} n^{-d}\lambda(C_n) = \lim_{n\to\infty} n^{-d}\lambda(D_n) = \lambda(B).$$

The lower estimate of $n^{-d}F_{nB}$ is fairly easy: In view of (4), we have, for $n \geq 2Mk$,

$$\begin{aligned} n^{-d}F_{nB} &\geq n^{-d}F_{C_n} \geq n^{-d} \sum_{u \in K_n} F_{[Mu, M(u+e))} \\ &\geq n^{-d} \sum_{u \in K_n} \sum_{v < me} k^d \bar{F}_{ke} \circ \tau_{Mu} \circ \tau_{kv} \\ &\geq n^{-d}k^d m^d \operatorname{card}(K_n)(\bar{f} - \eta) - b^* \quad \text{a.e.,} \end{aligned}$$

where

$$b^* = \sup_{n \geq 2Mk} n^{-d} \sum_{u < ([n/k]+1)e} k^d |b| \circ \tau_{ku}.$$

As $n \geq 2Mk$ implies $[n/k]+1 \leq 2n/k$, we obtain, by the d-dimensional maximal inequality (Corollary 6.2.7 in [7])

$$P(b^* > \varepsilon) \leq \frac{4^d}{\varepsilon} \| b \|_1 \leq 4^d \varepsilon.$$

Since $n^{-d} k^d m^d \operatorname{card}(K_n) \to \lambda(B)$, this suffices to prove

$$\liminf n^{-d} F_{nB} - \lambda(B) \bar{f} \geq 0 \quad \text{a.e.}$$

For the upper estimate we introduce a new process. For $A \in \mathcal{V}$ let $A(k)$ be the union of all intervals $[ku, k(u+e))$ with $u \in A$. The process $G = \{G_A : A \in \mathcal{V}\}$ is defined by

$$G_A = F_{A(k)} - \sum_{u \in A} F_{[ku, k(u+e))}.$$

It is nonnegative and superadditive for the semigroup $\{\sigma_u : u \in V\}$ with $\sigma_u = \tau_{ku}$. Its spatial constant is, by our choice of k,

$$\gamma(G) = k^d \gamma(F) - \int F_{[0,ke)} \, \mathrm{d}P < k^d \varepsilon^2.$$

If J_n denotes the set of all $w \in V$ of the form $mu + v$ with $u \in L_n$, $v \in V$ and $v < me$, then D_n is the union of all intervals $[kw, k(w+e))$ with $w \in J_n$. We have

$$n^{-d} F_{nB} \leq n^{-d} F_{D_n} = n^{-d} \sum_{u \in L_n} \sum_{v < me} k^d \bar{F}_{ke} \circ \tau_{Mu} \circ \tau_{kv} + n^{-d} G_{J_n}.$$

The first term on the right-hand side is handled as above. The term $n^{-d} G_{J_n}$ can be estimated from above (for $n \geq 2Mk$) by $2^d k^{-d} G^*$ where

$$G^* = \sup_{\ell \geq 1} (2\ell)^{-d} G_{[0, 2\ell e)}.$$

(Replace J_n by the bigger set $[0, 2\ell e)$ for $\ell = [n/k]$. Then $n^{-d} \leq (k\ell)^{-d}$.) The maximal inequality of Akcoglu and Krengel [1, 7] yields

$$P(2^d k^{-d} G^* > \varepsilon) = P(G^* > 2^{-d} k^d \varepsilon) \leq 4^d k^{-d} \varepsilon^{-1} \gamma(G) \leq 4^d \varepsilon.$$

This suffices to prove

$$\limsup n^{-d} F_{nB} - \lambda(B) \bar{f} \leq 0 \quad \text{a.e.}$$

In these estimates the choice of B matters only for the convergence of $n^{-d} M^d \operatorname{card}(K_n)$ and $n^{-d} M^d \operatorname{card}(L_n)$ to $\lambda(B)$. The condition (1) guarantees that this convergence is uniform in $B \in \mathcal{B}$.

Remarks. (a) It does not seem difficult to extend the present uniform pointwise convergence to subadditive processes for a d-parameter semigroup of measure-preserving transformations in a σ-finite measure space. The L_1-mean convergence does not hold. The present argument applies on the maximal subset of Ω carrying a finite invariant measure. On the complement the limit is 0.

(b) It also seems possible to extend the result (2) in the *additive* case to semigroups of $L_1 - L_\infty$-contractions using the methods of Brunel [3]. However, so far the needed maximal estimate is not available for positive $L_1 - L_\infty$-contractions and superadditive processes. In the operator case it is also missing for increasing sequences $u(1) < u(2) < \cdots$ as in (3).

(c) The restriction to a fixed increasing sequence $u(1) < u(2) < \cdots$ can be dropped in (3) in the *additive* case by imposing the condition $F_{[0,e)} \in L \log^{d-1} L$; see Theorem 2.4 of Sucheston [10]. This then yields a strengthening of the ergodic theorem of Zygmund [12]; see also [6, § 6.1]) namely,

$$\sup\{|\lambda([0, u))^{-1} F_{uB} - \lambda(B)\bar{f}|: B \in \mathscr{B}\} \to 0 \quad \text{a.s.}$$

as $u \to \infty$, where $uB = \{(u_i b_i): B \in \mathscr{B}\}$.

3. Continuous parameter processes

Let $\mathscr{V}_R$ denote the family of bounded Borel sets in $V_R = (\mathbb{R}^+)^d$. The definition of stationarity and subadditivity is the same as in the discrete parameter case with $\mathscr{V}$ replaced by $\mathscr{V}_R$ and V by V_R. Now, however, it is unnecessary to extend the parameter set of a process $F = \{F_A: A \in \mathscr{V}_R\}$.

We shall make use of suprema over uncountable families of measurable functions. They are to be understood as "essential" suprema in the space of equivalence classes (mod null sets). $\sup\{f_i, i \in I\}$ is the unique minimal equivalence class h with $h \geq f_i$ for all i. It agrees with $\sup\{f_i, i \in I_0\}$ for suitable countable $I_0 \subset I$. Similarly $\leq$ means $\leq$ a.e.

Theorem 2. *Let $\mathscr{B}$ be as in Theorem* 1, *and let $F = \{F_A: A \in \mathscr{V}_R\}$ be a subadditive process with*

$$\sup\{|F_A|: A \subset [0, e), A \in \mathscr{V}_R\} \in L_1. \tag{5}$$

Then the assertion of Theorem 1 *remains true.*

Proof. By the stationarity and by (5) the equivalence classes

$$q_u := \sup\{|F_A|: A \subset [u, u+e)\}, \quad u \in V,$$

belong to L_1. The process $\{Q_A: A \in \mathscr{V}\}$ with $Q_A = \sum_{u \in A} q_u$ is a discrete parameter additive process.

We again consider the sets C_n, D_n defined in the proof of Theorem 1. As the sets C_n, D_n are finite disjoint unions of intervals $[u, u+e)$ with $u \in V$, the proof of the discrete parameter theorem yields

$$|n^{-d}F_{C_n} - \lambda(B)\bar{f}| \to 0 \quad \text{and} \quad |n^{-d}F_{D_n} - \lambda(B)\bar{f}| \to 0 \quad \text{a.e.}$$

(uniformly in B). Although F is not nonnegative and superadditive, the discrete parameter process

$$F_A^* = F_{\cup\{[u,u+e):u\in A\}}, \quad A \in \mathcal{V},$$

is a linear combination of nonnegative superadditive processes. Applying the proof of Theorem 1 to the process $Q = \{Q_A\}$ we obtain

$$n^{-d}Q_{D_n\setminus C_N} \to 0 \quad \text{a.e.} \quad (\text{uniformly in } B).$$

The subadditivity of F yields

$$F_{nB} \leq F_{C_n} + F_{nB\setminus C_n} \leq F_{C_n} + Q_{D_n\setminus C_n}$$

and

$$F_{D_n} \leq F_{nB} + F_{D_n\setminus nB}.$$

Hence $F_{D_n} - F_{C_n} \leq F_{D_n\setminus nB} + Q_{D_n\setminus C_n}$. On the other hand $F_{D_n\setminus nB} \leq Q_{D_n\setminus C_n}$. Together these estimates imply

$$|n^{-d}F_{nB} - \lambda(B)\bar{f}| \to 0 \quad \text{a.e.}$$

uniformly for $B \in \mathcal{B}$. □

Remarks. (a) A stationary point process, for which the number of points in $[0, e)$ is integrable, is an example of a process satisfying the assumptions of the theorem.

(b) It again seems easy to prove the extension to σ-finite measure spaces and the generalization in which the sequence ne is replaced by an increasing sequence in V.

(c) Nguyen–Zessin [8] have employed the condition (4) for convex sets to prove a pointwise ergodic theorem for *additive* processes and *convex* sets.

(d) The condition (5) above is analogous to the condition used by Kingman [6] in the continuous parameter case.

4. A counterexample

The proof in Bass and Pyke [2] of the uniform strong law for i.i.d. arrays depends only on the ability to approximate the sets in $\mathcal{B}$ by a finite number of inner and outer approximating sets. Condition (1) makes this possible by permitting the use of rectilinear sets for these approximations. Recently, Giné and Zinn [5] introduced the appropriate "metric-entropy" conditions and used these to obtain complete characterizations of the uniform strong law in the additive i.i.d. case. Roughly speaking, a finite metric entropy condition permits finite approximations by preventing $\mathcal{B}$ from containing too many sets which look very different. This is the case if (1) holds, but in contrast to (1), a finite metric entropy assumption does not impose any condition on the sets in $\mathcal{B}$ if $\mathcal{B}$ is a *finite* family.

We now show by example that such a weaker condition is not sufficient in the present ergodic theoretic setting even for $d=1$. Take

$$\Omega=\{0,1\}, \quad P(0)=P(1)=\tfrac{1}{2}, \quad \tau 0=1, \quad \tau 1=0,$$

and let

$$f(0)=1, \qquad f(1)=-1.$$

Put

$$F_A=\sum_{i\in A\cap V} f\circ\tau^i.$$

We construct a single set $B\subset[0,1)$ such that $n^{-1}F_{nB}$ does not converge.

Let $p_1<p_2<\cdots$ be an increasing sequence of prime numbers. For even i, let

$$B_i=\{2k/p_i\colon 0<k<p_i/2\}.$$

For odd i, let

$$B_i=\{(2k-1)/p_i\colon 0<k<p_i/2\}.$$

Let B be the disjoint union of the sets B_i. If $n=p_i$ and i is even, then $nB\cap\mathbb{N}$ consists of the even natural numbers smaller than p_i. Hence $n^{-1}F_{nB}\approx f/2$ if n is large. If $n=p_i$ and i is odd, then $nB\cap\mathbb{N}$ consists of the odd natural numbers smaller than p_i-1. Hence

$$n^{-1}F_{nB}\approx f\circ\tau/2.$$

Since $f\neq f\circ\tau$, the sequence $n^{-1}F_{nB}$ cannot converge.

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